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Cycles in the cube-connected cycles graph ☆

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Abstract

In this paper we study the existence of cycles of all lengths in the cube-connected cycles graph and we establish that this graph is no far from being pancyclic in case n odd and bi-pancyclic in case n even. © 1998 Elsevier Science B.V. All rights reserved.

1. Introduction

In this paper we study the existence of cycles of given lengths in the cube-connected cycles graph CCC_n (see definition in Section 2). This graph was introduced by Preparata and Vuillemin in 1981 ([6]) as a good alternate for the hypercube, having a fixed degree equal to 3 and yet a small diameter compared to its number of vertices. This graph was since then studied by a lot of people, who showed in particular, that it also has good properties as far as communications are concerned. It was proved that CCC_n contains a hamiltonian cycle, which is an interesting property to realise distributed algorithms. However, its cycle structure is still not completely known, as to whether it contains cycles of any given length. For n even, this graph is bipartite. For any $n, n \geqslant 2$, CCC_n is a Cayley graph on the wreath product of $Z/2Z$ by Z/nZ [1]. Stong has given a general construction for hamiltonian cycles in the case of Cayley graphs on wreath product groups [8]. Rosenberg adapted the proof to CCC_n [7]. He also proved the existence of some families of cycles in CCC_n of length $n2^k - c(n-2)$ (see Theorem 4.1 in Section 4). The problem of the existence of cycles of all possible lengths remains open (i.e. cycles of all even length if n is even).

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In this paper we establish that the graph CCC_n is not far from being pancyclic in case n odd and even pancyclic (i.e. containing cycles of all even lengths) in case n even. More precisely we prove the following result.

Theorem 1.1. *The cube-connected cycles graph CCC_3 contains cycles of length 3 and all lengths between 8 and $3 \times 2^3 = 24$. CCC_4 contains cycles of length 4 and cycles of all even lengths between 8 and $4 \times 2^4 = 64$. For $n \geqslant 5$, CCC_n contains cycles of length n and of every even length between 8 and $n2^n$ except 10 and possibly $n2^n - 2$. Furthermore, for n odd, CCC_n contains cycles of every odd length between $n+6$ and $n2^n - n - 2$, and also cycles of length $n2^n - n + 2$.*

The proof is divided into three parts corresponding to different sets of values of the lengths l of the cycles, and different methods to prove the existence of such cycles. First, in Section 3, using the hamiltonian cycle of the hypercube $H(n)$ given by the reflected Gray code, we directly construct cycles of lengths l, for values of l roughly between 3×2^n and $(n-1) \cdot 2^n$. Then in Section 4, beginning with a construction of cycles of CCC_n given by Rosenberg [7] and deleting successively suitable vertices, we construct cycles of lengths l for the highest values of l. Finally, in Section 5, for $n \geqslant 3$, we construct cycles of the smallest lengths by induction.

2. Notation and definition

We consider simple graphs and use common terminology (for example, see [2]). Let $G = (V(G), E(G))$ be a graph, with vertex set $V(G)$ and edge set $E(G)$. We denote by $[u, v]$, $u \in V$, $v \in V$, the edge of G with ends u, v. We denote a path by the sequence of its vertices, for example $[u_0, u_1, \ldots, u_k]$. The length of a path is as usual the number of edges of the path (k in our example).

Binary strings are used as vertex labels or components of vertex labels for the classes of graphs studied in this paper. For a binary string $x = x_0 x_1 \cdots x_i \cdots x_{n-1}$ position 0 corresponds to the leftmost bit, position $n-1$ to the rightmost bit. We use $x(i)$ or $x_0 x_1 \cdots \overline{x_i} \cdots x_{n-1}$ to denote the binary string obtained from $x = x_0 x_1 \cdots x_i \cdots x_{n-1}$ by complementing the bit in position i. Similarly, $x(i,j)$ and $x_0\, x_1 \cdots \overline{x_i} \cdots \overline{x_j} \cdots x_{n-1}$ mean x where bits i and j are complementary, and so on. We use i^j to denote a string of j i's. We will abbreviate "the vertex with label x" to "vertex x".

We denote by $H(n)$ the n-dimensional binary hypercube. The graph $H(n)$ has 2^n vertices which are labelled with the binary strings of length n. Two vertices of $H(n)$ are adjacent if and only if their labels differ in exactly one bit position.

Definition 2.1.

- *The graph CCC_n* has $n \times 2^n$ vertices, labelled (ℓ, x), where ℓ is an integer between 0 and $n-1$, called the *level* of the vertex,[1] and x is a binary string of length n,

[1] In this paper, all arithmetic on indices and levels concerning CCC_n is assumed to be modulo n.

called the *row* of x. Two vertices (ℓ,x) and (ℓ',y) are adjacent if and only if either $x=y$ and $|\ell-\ell'|=1$, or $\ell=\ell'$ and $y=x(\ell)$. In this last case, x and y differ in exactly the bit in position ℓ.

- *Level* ℓ of CCC_n is the set of all vertices with level ℓ. The edges joining two vertices of the same level are called *level-edges* or *H-edges*.
- *Row* x of CCC_n is the set of all vertices with row x. The edges that connect (ℓ,x) to its neighbours $(\ell+1,x)$ and $(\ell-1,x)$ are called *C-edges* and form a cycle of length n called the *fundamental cycle* $C(x)$ defined by x.

For $n=2$, CCC_n is simply the cycle of length 8. So in this article we will only consider the case $n\geqslant 3$.

3. Cycles of intermediate lengths

Let us recall that the *cube-connected cycles* graph of dimension n [6] is derived from $H(n)$ by replacing each vertex x of $H(n)$ by a cycle of length n. Thus, a natural idea to find cycles in CCC_n is to start from cycles of the hypercube. In this section, we construct cycles of intermediate lengths using a hamiltonian cycle of $H(n)$. Let us first recall the construction of the hamiltonian cycle of $H(n)$ induced by the symmetric Gray code. We denote by $\mathscr{C}_n$ the wrapped around sequence of vertices of this cycle. It is constructed in a recursive way as follows. Starting in $H(1)$ with the sequence 0, 1, the sequence $\mathscr{C}_n$ of vertices of $H(n)$ is obtained from the sequence $\mathscr{C}_{n-1}$ of vertices of $H(n-1)$ by writing the vertices of $\mathscr{C}_{n-1}$ prefixed by 0 and then by the reverse of $\mathscr{C}_{n-1}$, the vertices being now prefixed by 1. This can be written $\mathscr{C}_n=0\mathscr{C}_{n-1},1\overline{\mathscr{C}_{n-1}}$. For example,

- for $H(2)$ the sequence $\mathscr{C}_2$ is 00, 01, 11, 10;
- for $H(3)$ the sequence $\mathscr{C}_3$ is 000, 001, 011, 010, 110, 111, 101, 100;
- for $H(4)$ the sequence $\mathscr{C}_4$ is 0000, 0001, 0011, 0010, 0110, 0111, 0101, 0100, 1100, 1101, 1111, 1110, 1010, 1011, 1001, 1000.

Let x be any vertex of $H(n)$. We denote by x^+ and x^- the vertices following and preceding x, respectively, in the wrapped around sequence $\mathscr{C}_n$ and we denote by $d(x)$ the position of the bit on which x and x^+ differ. Let us recall that the positions of the bits of a vertex x are numbered from 0 on the left to $n-1$ on the right. We associate to the vertex x of $H(n)$ the two vertices of CCC_n defined by $ant(x)=(d(x^-),x)$ and $succ(x)=(d(x),x)$. For example, in the case of $H(4)$, if $x = 0111$, then $x^+=0101$, $x^-=0110$, $d(x) = 2$, $ant(x)=(3,0111)$ and $succ(x)=(2,0111)$. Thus for any vertex x of $H(n)$,

- the vertices $succ(x)=(d(x),x)$ and $ant(x^+)=(d(x),x^+)$ are adjacent in CCC_n through an H-edge in level $d(x)$,
- the vertices $ant(x)=(d(x^-),x)$ and $succ(x)=(d(x),x)$ are on the same row, so that they can be joined, in CCC_n, by two different paths along the fundamental cycle $C(x)$ defined by x, the sum of the lengths of these two paths being equal to n.

In order to get cycles of CCC_n, respecting the order of the vertices in the sequence $\mathscr{C}_n$, we construct a new sequence by replacing each vertex x in $\mathscr{C}_n$ by the sequence of vertices of CCC_n composed of $ant(x)$, $\overline{P_x}$, $succ(x)$, where P_x is one of the two paths of the fundamental cycle $C(x)$ joining $ant(x)=(d(x^-),x)$ to $succ(x)=(d(x),x)$, and $\overline{P_x}$ the set of intermediate vertices of P_x. For example, for $n=2$, from $\mathscr{C}_2=\{00,01,11,10\}$ we get $\{(0,00),(1,00),(1,01),(0,01),(0,11),(1,11),(1,10),(0,10)\}$. In this example the paths P_x are reduced to an edge and $\overline{P_x}$ is empty. In the general case, the length of the new cycle is equal to 2^{n+1} plus the sum of the number of intermediate vertices of the paths P_x, i.e. $|d(x)-d(x^-)|-1$ or $n-1-|d(x)-d(x^-)|$. All these numbers are taken modulo n.

In the following lemmata we study the sequence of the values $d(x)$, $x\in\mathscr{C}_n$, and the associated sequence $|d(x^+)-d(x)|$, $x\in\mathscr{C}_n$, in order to calculate the lengths of the cycles of CCC_n (note that, since we will be interested in the sum of these values over all x in $\mathscr{C}_n$, this is the same as to study the sequence of $|d(x)-d(x^-)|$).

Lemma 3.1. *Let S_n denote the sequence of values $d(x)$ for the vertices x of $\mathscr{C}_n$ and S'_n be the sequence obtained from S_n by deleting the last element. Then*

(i) *For $n=2$, $S_2=\{1,0,1,0\}$, $S'_2=\{1,0,1\}$.*

(ii) *For $n\geqslant 3$, $S_n=\{S'_{n-1}+1,0,S'_{n-1}+1,0\}$, where $S'_{n-1}+1$ denotes the sequence obtained from S'_{n-1} by adding 1 to each element.*

(iii) *For $n\geqslant 2$, each element of S_n in odd position is equal to $n-1$.*

Proof. The proof is obvious by the inductive construction of $\mathscr{C}_n=0\mathscr{C}_{n-1},1\overline{\mathscr{C}_{n-1}}$. Having constructed the sequence S_{n-1}, we obtain the sequence S_n as follows: take the $n-2$ first terms of S_{n-1} and add 1 to each of them (since we have introduced a 0 on the left of the strings, and then we have increased by one the "position of the bits" used for $d(x)$), then take 0 (the corresponding vertices of $H(n)$ differ in the first-left position), and repeat twice the modified sequence. □

Lemma 3.2. *Let D_n denote the sequence of the 2^n differences $|d(x^+)-d(x)|$, $x\in\mathscr{C}_n$. Then the following holds:*

(i) *For $n=2$, $D_2=\{1,1,1,1\}$.*

(ii) *For $n\geqslant 3$, $D_n=\{D''_{n-1},n-1,n-1,D''_{n-1},n-1,n-1\}$, where D''_{n-1} is the subsequence of D_{n-1} containing the $2^{n-1}-2$ first elements.*

(iii) *D_n contains 2^{n-i} values equal to i for $1\leqslant i\leqslant n-2$ and 4 values equal to $n-1$.*

Proof. Since $S_n=\{S'_{n-1}+1,0,S'_{n-1}+1,0\}$, the second part of the sequence D_n repeats the first one. The $2^{n-1}-2$ first elements of D_n are equal to the $2^{n-1}-2$ first elements of D_{n-1}. Since S'_{n-1} begins and ends by $n-2$, the $2^{n-1}-1$ and 2^{n-1}th elements of D_n are $n-1$. Part (iii) is easy by induction. □

To prove Proposition 3.6 we need some technical lemmata.

Lemma 3.3. *Let $q_i, 1 \leqslant i \leqslant k$, be given positive integers and consider the set of integers $E = \{\sum_{i=1}^{k} ip_i,\ 0 \leqslant p_i \leqslant q_i\}$. Then $E = \{0, 1, \ldots, \sum_{i=1}^{k} iq_i\}$ (i.e. E contains all integers from 0 to $\sum_{i=1}^{k} iq_i$).*

Proof. The proof is by induction on k. It is clear for $k = 1$. Assume

$$E' = \left\{ \sum_{i=1}^{k-1} ip_i,\ 0 \leqslant p_i \leqslant q_i \right\} = \left\{ 0, 1, \ldots, \sum_{i=1}^{k-1} iq_i \right\}.$$

Let us consider some value of r such that $0 \leqslant r \leqslant \sum_{i=1}^{k} iq_i$. If $r \geqslant kq_k$, we consider $r' = r - kq_k$, so that $r' \leqslant \sum_{i=1}^{k-1} iq_i$. By the induction hypothesis, $r' = \sum_{i=1}^{k-1} ip_i$ and thus $r = \sum_{i=1}^{k-1} ip_i + kq_k$. If $r < kq_k$ the euclidean division of r by k gives $r = kp_k + r_k$ with $r_k = 0$ or $r_k < k$ and $p_k < q_k$. Then $r = s + kp_k$ for some $s, 1 \leqslant s \leqslant k-1$ and $p_k < q_k$, and this finishes the proof by taking $p_s = 1$ and $p_i = 0$ for $i \neq k,\ s$. □

As a corollary of Lemma 3.3 we obtain the following result.

Lemma 3.4. *Let $q_i, 1 \leqslant i \leqslant k$, be given positive integers and let us consider the set of integers $E = \{\sum_{i=1}^{k} 2ip_i,\ 0 \leqslant p_i \leqslant q_i\}$. Then $E = \{0, 2, \ldots, 2\sum_{i=1}^{k} iq_i\}$ (i.e. E contains all even integers between 0 and $2\sum_{i=1}^{k} iq_i$).*

In case of odd terms we get the following result.

Lemma 3.5. *Let q_i, $1 \leqslant i \leqslant k$, be given positive integers with $q_0 \geqslant 2$, and let us consider the set of integers $E = \{\sum_{i=0}^{k} (2i+1)p_i,\ 0 \leqslant p_i \leqslant q_i\}$. Then $E = \{0, 1, \ldots, \sum_{i=1}^{k} (2i+1)q_i\}$ (i.e. E contains all integers between 0 and $\sum_{i=1}^{k} (2i+1)q_i$).*

Proof. The proof is quite similar to the proof of Lemma 3.3, so we omit it. □

The construction given above allows us to prove the existence of cycles of intermediate lengths in $CCC(n)$ as specified in the next theorem.

Proposition 3.6. *For n even, CCC_n contains cycles of all even lengths between $l_{\text{min-ev}} = 3 \times 2^n - 2^{n/2+2} + 4$ and $l_{\text{max-ev}} = (n-1)2^n + 2^{n/2+2} - 4$. For n odd, CCC_n contains cycles of all lengths between $l_{\text{min-od}} = 3 \times 2^n - 3 \times 2^{(n+1)/2} + 4$ and $l_{\text{max-od}} = (n-1)2^n + 3 \times 2^{(n+1)/2} - 4$.*

Proof. As seen previously, the difference $i = |d(x^+) - d(x)|$ for $x \in \mathscr{C}_n$ enables us to construct two different paths P_x using C-edges, one containing $i-1$ intermediate vertices, the other containing $n-1-i$ intermediate vertices.

By Part (iii) of Lemma 3.2 there exist 2^{n-i} paths P_x, for $1 \leqslant i \leqslant n-2$, containing either $i-1$ or $n-1-i$ intermediate vertices and 4 paths P_x containing either 0 or

$n-2$ intermediate vertices. Thus, the number of paths P_x containing either $i-1$ or $n-1-i$ intermediate vertices is equal to

- $2^{n-i}+2^i$, for $2 \leqslant i < \lfloor n/2 \rfloor$,
- $2^{n-i}+2^i$, for n odd and $i = \lfloor n/2 \rfloor$,
- $2^{n/2}$, for n even and $i = n/2$,
- $2^{n-1}+4$ for $i=1$.

For a given choice of paths P_x, let us denote by p_i, $1 \leqslant i \leqslant \lfloor n/2 \rfloor$, the number of paths P_x containing $i-1$ intermediate vertices in the constructed cycle of CCC_n. We split the proof into two cases, depending on the parity of n.

Let us recall that the length of a cycle of CCC_n we have constructed is equal to twice the number of vertices of the hamiltonian cycle of $H(n)$ we started from, i.e. 2×2^n, plus the sum of the number of intermediate vertices in the paths P_x for all x in $\mathscr{C}_n$.

Case 1: *n odd*. By considering all the possible choices of paths P_x we see that in CCC_n we can obtain cycles of lengths l with

$$l = 2 \times 2^n + (2^{n-1}+4-p_1)(n-2) + \sum_{i=2}^{\lfloor n/2 \rfloor} [p_i(i-1) + (2^{n-i}+2^i-p_i)(n-1-i)],$$

$$l = \left[2 \times 2^n + (2^{n-1}+4)(n-2) + \sum_{i=2}^{\lfloor n/2 \rfloor} (2^{n-i}+2^i)(n-1-i) \right] - \left[p_1(n-2) + \sum_{i=2}^{\lfloor n/2 \rfloor} p_i(n-2i) \right],$$

for all $0 \leqslant p_1 \leqslant 2^{n-1}+4$ and $0 \leqslant p_i \leqslant 2^{n-i}+2^i$, $2 \leqslant i \leqslant \lfloor n/2 \rfloor$.

From this we deduce that the minimum length $l_{\text{min-od}}$ of this set of cycles is obtained for $p_1 = 2^{n-1}+4$ and $p_i = 2^{n-i}+2^i$, $2 \leqslant i \leqslant \lfloor n/2 \rfloor$, i.e., $l_{\text{min-od}} = 3 \times 2^n - 3 \times 2^{(n+1)/2} + 4$. We deduce also that the maximum length of this set of cycles is obtained for $p_i = 0, 1 \leqslant i \leqslant \lfloor n/2 \rfloor$, and is $l_{\text{max-od}} = (n-1)2^n + 3 \times 2^{(n+1)/2} - 4$.

To end the proof we only need to prove that, in fact, we obtain every intermediate value between $l_{\text{min-od}}$ and $l_{\text{max-od}}$. Since $n-2i$ is odd, we can write the following equality.

$$\sum_{i=1}^{\lfloor n/2 \rfloor} p_i(n-2i) = \sum_{j=0}^{(n-3)/2} (2j+1) p_{(n-2j-1)/2}.$$

Using Lemma 3.5, we see that we obtain any intermediate value between $l_{\text{min-od}}$ and $l_{\text{max-od}}$.

Case 2: *n even*. The proof is almost the same as in the case n is odd. CCC_n contains cycles of lengths

$$l = 2 \times 2^n + (2^{n-1} + 4 - p_1)(n-2) + 2^{n/2}(n-2)/2 + \sum_{i=2}^{n/2-1} [p_i(i-1) + (2^{n-i} + 2^i - p_i)(n-1-i)],$$

$$l = \left[2 \times 2^n + (2^{n-1} + 4)(n-2) + 2^{n/2}(n-2)/2 + \sum_{i=2}^{n/2-1} (2^{n-i} + 2^i)(n-1-i)\right] - \left[p_1(n-2) + \sum_{i=2}^{n/2-1} p_i(n-2i)\right],$$

for all $0 \leqslant p_1 \leqslant 2^{n-1} + 4$ and $0 \leqslant p_i \leqslant 2^{n-i} + 2^i$, $2 \leqslant i < n/2$. From this we deduce that the minimum length $l_{\min-ev}$ of this set of cycles is obtained for $p_1 = 2^{n-1} + 4$ and $p_i = 2^{n-i} + 2^i$, $2 \leqslant i \leqslant n/2 - 1$, i.e., $l_{\min-ev} = 3 \times 2^n - 2^{n/2+2} + 4$. We deduce also that the maximum length of this set of cycles is obtained for $p_i = 0$ and is $l_{\max-ev} = (n-1)2^n + 2^{n/2+2} - 4$.

Since $n - 2i$ is even we have

$$\sum_{i=2}^{n/2-1} p_i(n-2i) = \sum_{j=1}^{(n-4)/2} 2j\, p_{(n-2j)/2}.$$

We use Lemma 3.4 to get every even value between $l_{\min-ev}$ and $l_{\max-ev}$ and, thus, to finish the proof. □

4. Existence of long cycles

Let us first introduce some notation specific to this part. In the notation of a vertex (ℓ, x), where x is a binary string of length n, we may need to specify only some of the bits, in which case for example $0^k u$ will be the binary string of length n with the first k bits equal to 0 and any binary string u of length $n - k$. Similarly, $0^k 101\vec{0}$ will stand for $0^k 1010^{n-k-3}$. More generally, $\vec{0}$ will stand for a string of zeros of necessary length to complete the length of the total string to n.

To prove the existence of long cycles, we first need to give a contruction due to Rosenberg [7], since the proof of our result is based on it. Part of the proof of the following proposition has been taken from Rosenberg's report, with the necessary changes. We will use this result later.

Proposition 4.1. *For all n, the cube-connected cycles graph CCC_n contains a cycle of length l for the following values of l*:

- $l = n$,
- $l = n2^k - (n-2)c$ *for* $k = 2$ *or* 3 *and* $0 \leqslant c \leqslant 2^k$,

- *$l = n2^k - (n-2)c$ for $4 \leqslant k \leqslant n$ and $0 \leqslant c \leqslant 2^k - 2^{k-2-(k \bmod 2)}$.*

In particular, CCC_n is hamiltonian.

Proof. The proof first concentrates on the case $c = 0$ in the expression for the length l for the contained cycle. Hence, we wish to establish the containment in CCC_n of cycles of lengths $n2^k$ for all $k \in Z_{n+1} - \{1\}$.

The proof is an adaptation of the one from [8] that establishes that CCC_n has a hamiltonian cycle, by showing that every graph in the following family of subgraphs of CCC_n has a hamiltonian cycle. For $k \in Z_{n+1}$, the graph $CCC_n^{(k)}$ is the maximal connected component containing vertex $(0, \vec{0})$ of the induced subgraph of CCC_n on the set of vertices

$$V_k =_{\text{def}} Z_n \times \{x0^{n-k} \colon x \in Z_2^k\}.$$

Note that $CCC_n^{(k)}$ is obtained from CCC_n by deleting all level-edges of CCC_n at levels $k, k+1, \ldots, n-1$ and then deleting all vertices and edges that are no longer accessible from vertex $(0, \vec{0})$; in particular, $CCC_n^{(0)}$ is (isomorphic to) the n-vertex cycle C_n, and $CCC_n^{(n)}$ is identical to CCC_n. We let $CCC_n^{(k)}$ inherit a level structure in the natural way from CCC_n.

We establish that every graph $CCC_n^{(k)}$ has a hamiltonian cycle, by induction on k, with two base cases. The base case $CCC_n^{(0)}$ will yield the desired result for all even values of k; the base case $CCC_n^{(3)}$ will yield the desired result for all odd values of$k$.

Lemma 4.2. *For all $k \in Z_{n-1} - \{1\}$, if $CCC_n^{(k)}$ is hamiltonian, then so also is $CCC_n^{(k+2)}$.*

Proof. An illustration of the construction can be seen in Fig. 2, where a hamiltonian cycle in CCC_n^5 is shown, constructed from four copies of a hamiltonian cycle in CCC_n^3. The figure is given with $n = 5$, but could easily be extended by straight lines to the right in each row. Assume for induction that we are given a hamiltonian cycle $\mathscr{C}$ in $CCC_n^{(k)}$. We extend the induction by traversing the hamiltonian cycle in $CCC_n^{(k)}$ repeatedly. As an aid in describing the multiple traversals, we say that a traversal *proceeds up* the cycle when it proceeds along the cycle in the increasing order of the levels of its vertices, $j, j+1, j+2, \ldots$ and that the traversal *proceeds down* the cycle when it proceeds along the cycle in the order $j, j-1, j-2, \ldots$ of the levels of its vertices, all addition being modulo $n2^k$.

Implicit in the formula for pruning CCC_n to produce $CCC_n^{(k)}$ is the fact that one can construct $CCC_n^{(k+2)}$ by taking four copies of $CCC_n^{(k)}$, call them Copies 00, 10, 01, and 11, and interconnecting them so as to obtain a copy of $CCC_n^{(k+2)}$. The interconnection begins with a renaming of the row strings of the vertices of $CCC_n^{(k)}$ as indicated in the following table.

In $CCC_n^{(k)}$	In Copy 00	In Copy 10	In Copy 01	In Copy 11
$x0^{n-k}$	$x0^{n-k}$	$x100^{n-k-2}$	$x010^{n-k-2}$	$x110^{n-k-2}$

Now interconnect the four copies by adding to them the level-k and level-$(k+1)$ edges of CCC_n in just the way that makes the resulting graph isomorphic to $CCC_n^{(k+2)}$.

One can now trace out a hamiltonian cycle in $CCC_n^{(k+2)}$ as follows. We refer freely to the four copies of $CCC_n^{(k)}$ that comprise $CCC_n^{(k+2)}$.

1. Start at vertex $(k,\ \vec{0})$ in Copy 00 of $CCC_n^{(k)}$, and proceed up its hamiltonian cycle until vertex $(k+1,\ \vec{0})$.
2. Cross from vertex $(k+1,\ \vec{0})$ in Copy 00 to vertex $(k+1,\ 0^k01\vec{0})$ in Copy 01.
3. Starting at vertex $(k+1,\ 0^k01\vec{0})$ in Copy 01, proceed down its hamiltonian cycle until vertex $(k,\ 0^k01\vec{0})$.
4. Cross from vertex $(k,\ 0^k01\vec{0})$ to vertex $(k,\ 0^k11\vec{0})$ in Copy 11.
5. Starting at vertex $(k,\ 0^k11\vec{0})$ in Copy 11, proceed up its hamiltonian cycle until vertex $(k+1,\ 0^k11\vec{0})$.
6. Cross from vertex $(k+1,\ 0^k11\vec{0})$ to vertex $(k+1,\ 0^k10\vec{0})$ in Copy 10.
7. Starting at vertex $(k+1,\ 0^k10\vec{0})$ in Copy 10, proceed down its hamiltonian cycle until vertex $(k,\ 0^k10\vec{0})$.
8. Cross from vertex $(k,\ 0^k10\vec{0})$ to vertex $(k,\ \vec{0})$ in Copy 00.

We claim that the above procedure does, indeed, specify a walk within $CCC_n^{(k+2)}$, i.e. that every prescribed step of the walk crosses just one edge of the graph. A facet of this claim that is not completely evident resides in our procedure's implicit exploitation of the following property.

Property 1. *For all k, every pair of vertices, (ℓ,x) and $(\ell+1,x)$, where $\ell \in Z_n - Z_k$ and $x \in Z_2^n$, appear consecutively in the hamiltonian cycle for $CCC_n^{(k)}$ that our procedure produces.*

Property 1 assures us that the pairs of vertices, (ℓ,x) and $(\ell+1,x)$, can be used to interconnect cycles in copies of $CCC_n^{(\ell)}$ in the way mandated by our procedure. One verifies the property by noting that at the point when our procedure produces a hamiltonian cycle for $CCC_n^{(k)}$, the vertices of interest are of degree 2 in $CCC_n^{(k)}$, hence must appear consecutively in the cycle.

The proof of Lemma 4.2 is now complete: the walk specified by our procedure interconnects the hamiltonian cycles in the four copies of $CCC_n^{(k)}$ in a way that yields a hamiltonian cycle in $CCC_n^{(k+2)}$. □

The proof of Proposition 4.1 when $c=0$ is completed by establishing the base cases of the induction.

Lemma 4.3. *Both $CCC_n^{(0)}$ and $CCC_n^{(3)}$ are hamiltonian.*

Proof. Because $CCC_n^{(0)}$ is (isomorphic to) the n-vertex cycle $\mathscr{C}_n$, it is hamiltonian. Let us concentrate, therefore, on finding a hamiltonian cycle in $CCC_n^{(3)}$ (see Fig. 1).

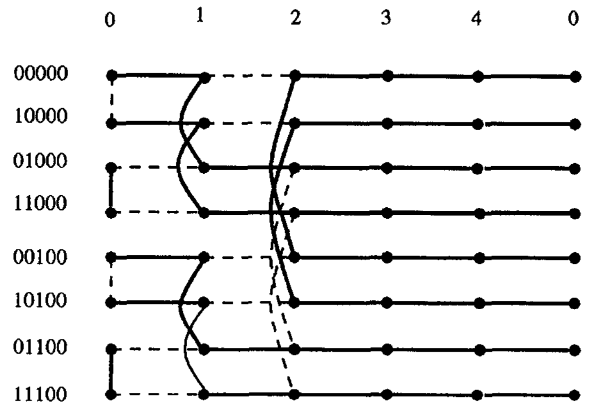

Fig. 1. A hamiltonian cycle in $CCC_5^{(3)}$.

We produce a hamiltonian cycle in $CCC_n^{(3)}$ from copies of the hamiltonian cycle in $CCC_n^{(0)}$ in much the same way that we produced a hamiltonian cycle in $CCC_n^{(k+2)}$ from copies of a hamiltonian cycle in $CCC_n^{(k)}$ in Lemma 4.2, except that we need eight copies of the "seed" cycle here, as opposed to the four copies that sufficed there.

Let us take eight copies of the hamiltonian cycle in $CCC_n^{(0)}$, call them Copies $000, 001, \ldots, 111$. Note that the vertices of the cycle comprise the set $\{(\ell, \vec{0}) : \ell \in Z_n\}$. Relabel the row strings in all copies of the cycle so that the vertices of Copy $\alpha\beta\gamma$ of the cycle $(\alpha, \beta, \gamma \in Z_2)$ comprise the set $\{(\ell, \alpha\beta\gamma\vec{0}) : \ell \in Z_n\}$. Under this vertex labelling, the vertex set of $CCC_n^{(3)}$ is just the union of the vertex sets of the eight cycles, and the edges of $CCC_n^{(3)}$ are the union of the edges of the cycles, plus the H-edges in levels $0, 1, 2$ of CCC_n. Using the same notion of traversing a cycle by proceeding *up* the cycle or *down* the cycle as we used in Lemma 4.2, we can now specify a hamiltonian cycle in $CCC_n^{(3)}$ as follows:

1. Start at vertex $(1, 000\vec{0})$ and proceed down the cycle, until vertex $(2, 000\vec{0})$.
2. Cross from vertex $(2, 000\vec{0})$ to vertex $(2, 001\vec{0})$ in Copy 001.
3. Proceed from vertex $(2, 001\vec{0})$ in Copy 001 up the cycle, until vertex $(1, 001\vec{0})$.
4. Cross from vertex $(1, 001\vec{0})$ to vertex $(1, 011\vec{0})$ in Copy 011.
5. Proceed from vertex $(1, 011\vec{0})$ in Copy 011 up the cycle, until vertex $(0, 011\vec{0})$.
6. Cross from vertex $(0, 011\vec{0})$ to vertex $(0, 111\vec{0})$ in Copy 111.
7. Proceed from vertex $(0, 111\vec{0})$ in Copy 111 down the cycle, until vertex $(1, 111\vec{0})$.
8. Cross from vertex $(1, 111\vec{0})$ to vertex $(1, 101\vec{0})$ in Copy 101.
9. Proceed from vertex $(1, 101\vec{0})$ in Copy 101 down the cycle, until vertex $(2, 101\vec{0})$.
10. Cross from vertex $(2, 101\vec{0})$ to vertex $(2, 100\vec{0})$ in Copy 100.
11. Proceed from vertex $(2, 100\vec{0})$ in Copy 100 up the cycle, until vertex $(1, 100\vec{0})$.
12. Cross from vertex $(1, 100\vec{0})$ to vertex $(1, 110\vec{0})$ in Copy 110.
13. Proceed from vertex $(1, 110\vec{0})$ in Copy 110 up the cycle, until vertex $(0, 110\vec{0})$.
14. Cross from vertex $(0, 110\vec{0})$ to vertex $(0, 010\vec{0})$ in Copy 010.

15. Proceed from vertex $(0, 010\vec{0})$ in Copy 010 down the cycle, until vertex $(1, 010\vec{0})$.
16. Cross from vertex $(1,\ 010\vec{0})$ to vertex $(1,\ 000\vec{0})$ in Copy 000.

The described walk constitutes a hamiltonian cycle in $CCC_n^{(3)}$. The existence of this cycle establishes the base case of the induction. □

To extend our result to the entire claimed set of contained cycles, by allowing the constant c to assume the given nonzero values, we need the following property.

Property 2. *For all values of $x \in Z_2^n$, except 0^3u for $k>3$ odd, $u \in Z_2^{n-3}$, and 0^2u for $k>2$ even, $u \in Z_2^{n-2}$, whenever the walks that define the cycles of lengths $n2^k$ encounter a fundamental cycle $C(x)$ of CCC_n defined by a given row string x, they traverse all $(n-1)$ edges of $C(x)$ other than the one that connects two of its adjacent vertices $(\ell,\ x)$ and $(\ell+1,\ x)$ for some ℓ.*

If, for the appropriate values of x, we alter the walk so that in $C(x)$ it traverses only the edge connecting $(\ell,\ x)$ and $(\ell+1,\ x)$ instead of the other $n-1$ edges of $C(x)$, then the length of the entire traversed cycle is decreased by precisely $n-2$. Pruning the walk in this way for c row strings yields the generalized result where c can vary between 0 and $2^k - 2^{k-2}$ if k is even and $2^k - 2^{k-3}$ if k is odd.

Property 2 is easy to verify by first noticing that it is valid for all possible rows x if $k=2$ or 3 (see the given construction in $CCC_n^{(2)}$ and $CCC_n^{(3)}$). Then in the given procedure above, for $k \geqslant 2$, the walk in $CCC_n^{(k+2)}$ interconnects the hamiltonian cycles in the four copies of $CCC_n^{(k)}$ by breaking one edge in each of the four rows $0^k00\vec{0}$, $0^k10\vec{0}$, $0^k01\vec{0}$ and $0^k11\vec{0}$. Thus, the structure of the walks in the rows different from these is not changed, and in particular, the structure of the hamiltonian cycle of $CCC_n^{(k)}$ in rows other than 0^2u is never altered when k is even and the structure of the hamiltonian cycle of $CCC_n^{(k)}$ in rows other than 0^3u is never altered when k is odd. □

We will now show that we can decrease the length of each of the cycle obtained in the previous proposition by a multiple of 6, and then by 4, or 8. An illustration of the following lemma is given in Fig. 2.

Lemma 4.4. *For any n, $n \geqslant 3$, any k odd, $3 \leqslant k \leqslant n$, and any r, $1 \leqslant r \leqslant (k-1)/2 - 1$, the hamiltonian cycle of $CCC_n^{(k)}$ defined above contains*

(a) *a path of length 3 with extremities in level k : $(k,\vec{0}),(k-1,\vec{0}),(k-1,0^{k-1}1\vec{0})$, $(k,0^{k-1}1\vec{0})$,*

(b) *for $k \geqslant 5$, 2^{k-2r-2} paths of length 7 with extremities $(2r+1, 0^{2r}10u0^{n-k})$ and $(2r+1, 0^{2r}11u0^{n-k})$, where $u \in Z_2^{k-2r-2}$. Each path uses four C-edges between level $2r$ and $2r+1$ on the rows $0^{2r}00u0^{n-k}$, $0^{2r}10v0^{n-k}$, $0^{2r}01u0^{n-k}$ and $0^{2r}11u0^{n-k}$, two H-edges in level $2r$ and one H-edge in level $2r+1$.*

Proof. Part (a) of the lemma can be verified directly for $k=3$. $CCC_n^{(3)}$ contains the following path of length 3: $(3,\vec{0}),(2,\vec{0})),(2,001\vec{0}),(3,001\vec{0})$. Also, in the

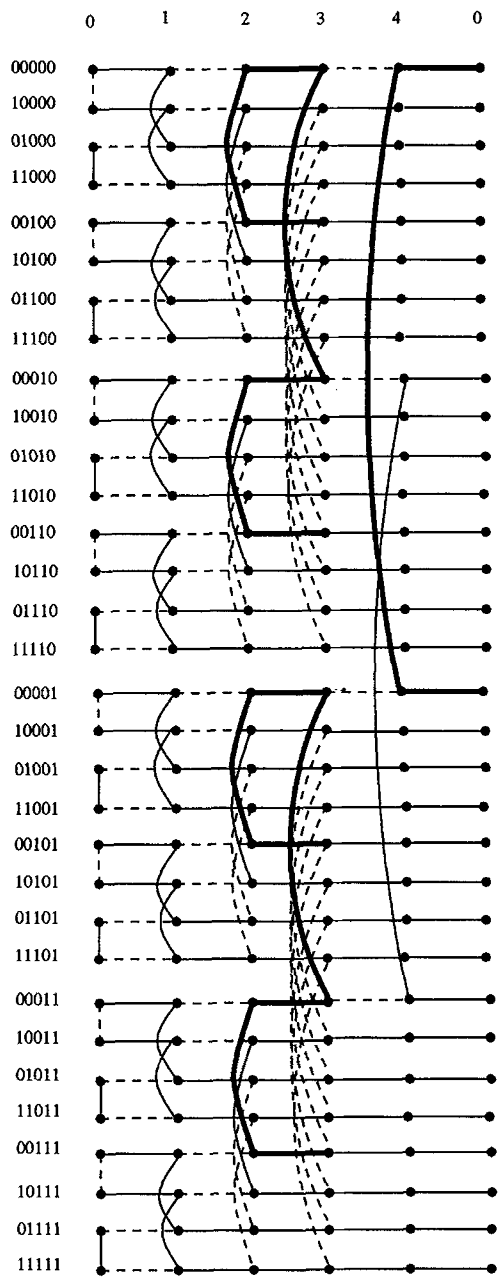

Fig. 2. Existence of paths of length 3 and 7 in CCC_5.

procedure for $k \geqslant 5$ to construct the hamiltonian cycle of $CCC_n^{(k)}$ from four copies of a hamiltonian cycle of $CCC_n^{(k-2)}$ we join the vertices $(k-1,\vec{0})$ of copy 00 and $(k-1, 0^{k-2}01\vec{0})$ of copy 01, which produces the expected path of length 3, $[(k,\vec{0}), (k-1,\vec{0}), (k-1,0^{k-2}01\vec{0}), (k,0^{k-2}01\vec{0})]$ in $CCC_n^{(k)}$.

The proof of part (b) uses induction on k. Let $k=2m+1$, $m \geqslant 2$.

Let us recall that, for $k \geqslant 5$, the construction of the hamiltonian cycle of $CCC_n^{(k)}$ uses four copies of a hamiltonian cycle of $CCC_n^{(k-2)}$. The procedure induces two paths of length 7 with extremities in level $k-2$. Indeed, we join the extremities $(k-2,\vec{0})$ and $(k-2,0^{k-2}10\vec{0})$ from the paths of length 3 specified in (a) (respectively, in copies 00 and 10 of $CCC^{(k-2)}$). This creates a path of length 7 with extremities $(k-2,0^{k-3}10\vec{0})$ and $(k-2,0^{k-3}11\vec{0})$. Similarly, we join the extremities $(k-2,0^{k-2}01\vec{0})$ and $(k-2,0^{k-2}11\vec{0})$ from the paths of length 3 specified in (a) (respectively, in copies 01 and 11 of $CCC_n^{(k-2)}$). This creates a second path of length 7 with extremities $(k-2,0^{k-3}101\vec{0})$ and $(k-2,0^{k-3}111\vec{0})$. In particular, starting from $CCC_n^{(3)}$, we obtain a hamiltonian cycle in $CCC_n^{(5)}$ which contains two paths of length 7 with extremities in level 3. Thus, part (b) of the lemma is satisfied for $k=5$. Assume now that $k \geqslant 7$, and that the result is true for $CCC_n^{(k')}$, $k' \leqslant k-2$.

Notice that the construction of the hamiltonian cycle of $CCC_n^{(k)}$ does not alter any of the edges of the hamiltonian cycles of the four copies of $CCC_n^{(k-2)}$ except for edges joining vertices between levels $k-2$ and $k-1$. Thus, the number of paths of length 7 of the type given in the lemma with extremities in levels $2r+1$ for r between 1 and $m-2$ is simply multiplied by 4 in the construction. And two new paths of length 7 appear between vertices of level $k-2$. □

The following proposition is an immediate consequence of Lemma 4.4.

Proposition 4.5. *For any $n \geqslant 5$ and any k odd, $5 \leqslant k \leqslant n$, $CCC_n^{(k)}$ and thus CCC_n contain cycles of all lengths l with $l = n2^k - 6\alpha$ for $0 \leqslant \alpha \leqslant (2^{k-2}-2)/3$.*

Proof. It is clear from the proof that the paths of length 7 from Lemma 4.4 are all vertex disjoint. Also, the extremities of any of these paths are adjacent in $CCC_n^{(k)}$. Thus, replacing any of these paths by the edge joining its extremities, we decrease the number of edges in the hamiltonian cycle of $CCC_n^{(k)}$ by 6. This can be done independently for all the paths. The total number of these paths of length 7 is equal to

$$\sum_{r=1}^{r=m-1} 2^{k-2r-2} = 2\sum_{s=0}^{s=m-2} 4^s = \frac{2^{k-2}-2}{3},$$

with $m=(k-1)/2$. □

The following lemma can be proved exactly as Lemma 4.4, and thus we omit the proof.

Lemma 4.6. *For any $n \geqslant 4$, any k even, $2 \leqslant k \leqslant n$, and any r, $1 \leqslant r \leqslant k/2-1$, the hamiltonian cycle of $CCC_n^{(k)}$ defined above contains*

(a) *a path of length 3 with extremities in level k: $(k,\vec{0}),(k-1,\vec{0}),(k-1,0^{k-1}1\vec{0})$, $(k,0^{k-1}1\vec{0})$,*

(b) *for $k \geqslant 4$, 2^{k-2r-1} paths of length 7 with extremities $(2r,0^{2r-1}10u0^{n-k})$ and $(2r,0^{2r-1}11u0^{n-k})$, where $u \in Z_2^{k-2r-1}$. Each path uses four C-edges between level $2r-1$ and $2r$ on the rows $0^{2r-1}00u0^{n-k}$, $0^{2r-1}10u0^{n-k}$, $0^{2r-1}01u0^{n-k}$ and $0^{2r-1}11u0^{n-k}$, two H-edges in level $2r$ and one H-edge in level $2r+1$.*

Counting the number of independent paths of length 7, each of which again can be replaced by an edge, we get the following.

Proposition 4.7. *For any n, $n \geqslant 4$, any k even, $4 \leqslant k \leqslant n$, $CCC_n^{(k)}$ and thus CCC_n contains cycles of all lengths l, with $l = n2^k - 6\beta$, for $0 \leqslant \beta \leqslant (2^{k-1}-2)/3$.*

Remark 4.8. Note that to produce the cycles in Proposition 4.5 we only altered edges from the original hamiltonian cycle in rows $00u$. Thus, using Property 2, we can still prune each of the new cycles by replacing a path of length $n-1$ by an edge in each of the $2^k - 2^{k-2}$ rows of labels starting with 01, 10 or 11 as we did before. Similarly, to produce the cycles in Proposition 4.7 we only altered edges from the original hamiltonian cycle in rows $0u$. Thus, using Property 2, we can still prune each of the new cycles by replacing a path of length $n-1$ by an edge in each of the 2^{k-1} rows of labels starting with 1.

Proposition 4.9. *For any n, $n \geqslant 5$, and any k odd, $5 \leqslant k \leqslant n$, $CCC_n^{(k)}$ and thus CCC_n contains cycles of all lengths l, with $l = n2^k - 4 - 6\beta$, for $0 \leqslant \beta \leqslant (2^{k-1}-4)/3$.*

Proof. In $CCC_n^{(k)}$ consider the two copies 0 and 1 of $CCC_n^{(k-1)}$. In each copy we take one of the cycles obtained in Corollary 4.7 and we delete the path of length 3 (which exists by Lemma 4.6) with extremities in level $k-1$. In $CCC_n^{(k)}$ we then join the remaining paths from the two copies with the H-edges of level $k-1$, $[(k-1,\vec{0}),(k-1,0^{k-1}1\vec{0})]$ and $[(k-1,0^{k-2}1\vec{0}),(k-1,0^{k-2}11\vec{0})]$. □

Proposition 4.10. *For any $n \geqslant 4$, and any k even, $4 \leqslant k \leqslant n$, $CCC_n^{(k)}$ and thus CCC_n contains cycles of all lengths l, with $l = n2^k - 4 - 6\alpha$, for $0 \leqslant \alpha \leqslant (2^{k-2}-4)/3$.*

Proof. Again, consider two copies 0 and 1 of $CCC_n^{(k-1)}$ in $CCC_n^{(k)}$. In each copy, we take either one of the cycles obtained in Corollary 4.5 for $k>4$, or the hamiltonian cycle of $CCC_n^{(3)}$ if $k=4$, and we delete the path of length 3 (which exists by Lemma 4.4) with extremities in level $k-1$. In $CCC_n^{(k)}$ we join the remaining paths from the two copies with the H-edges of level $k-1$, $[(k-1,\vec{0}),(k-1,0^{k-1}1\vec{0})]$ and $[(k-1,0^{k-2}1\vec{0}),(k-1,0^{k-2}11\vec{0})]$. □

Remark 4.11. Notice again that in the previous two propositions, for any $k \geqslant 4$, the construction does not alter rows beginning with 1.

Proposition 4.12. *For any n, $n \geqslant 4$, CCC_n contains cycles of all lengths l with $l = n2^n - 8 - 6\gamma$ for $0 \leqslant \gamma \leqslant 2^{n-3} - 2$.*

Proof. Let us first consider the case where n is odd. From Proposition 4.10, used with $k = n-1$, $CCC_n^{(n-1)}$ contains cycles of length $l' = n2^{n-1} - 4 - 6\alpha$ for $0 \leqslant \alpha \leqslant (2^{n-3} - 4)/3$. From Proposition 4.7, used with $k = n-1$, $CCC_n^{(n-1)}$ also contains cycles of length $l'' = n2^{n-1} - 6\beta$ for $0 \leqslant \beta \leqslant (2^{n-2} - 2)/3$. It is not difficult to verify, from the construction of the cycles, that each such cycle contains a path of length 3 with extremities in level $n-1$: $[(n-1, 010^{n-2}), (0, 010^{n-2}), (0, 110^{n-2}), (n-1, 110^{n-2})]$.

Consider the two copies 0 and 1 of $CCC_n^{(n-1)}$ contained in CCC_n, and take in one of them a cycle of length l' and in the other a cycle of length l''. Delete the path of length 3 mentioned above in each of these cycles, and join the two remaining paths in CCC_n using the H-edges in level $n-1$: $[(n-1, 010^{n-2}), (n-1, 010^{n-3}1)]$ and $[(n-1, 110^{n-2}), (n-1, 110^{n-3}1)]$. We obtain a path of length $l' + l'' - 4$. Thus CCC_n contains a cycle of length $n2^n - 8 - 6\gamma$ for $0 \leqslant \gamma \leqslant (2^{n-2} - 2)/3 + (2^{n-3} - 4)/3$. If n is even, the proof is exactly the same using Corollary 4.5 and Proposition 4.9 with $k = n-1$. □

Remark 4.13. Notice that in each of the two previous propositions the contruction has altered two and only two rows beginning with 1.

We conclude this part with the following result.

Proposition 4.14. *For any n, $n \geqslant 4$, CCC_n contains cycles of all even lengths l with $(n-1)2^n \leqslant l \leqslant n2^n$ except possibly $n2^n - 2$. Moreover, for any n odd, $n \geqslant 5$, CCC_n contains cycles of all odd lengths l' with $(n-1)2^n < l' \leqslant n2^n - (n-2)$ except possibly $n2^n - n$.*

Proof. The proof follows from what precedes, using the three remarks to prune the cycles obtained in the previous propositions in order to decrease their lengths by a multiple of $n-2$ as we did earlier. Together this implies that CCC_n contains cycles of lengths

$$n2^n - c(n-2) - 6\alpha - \varepsilon 4 - \varepsilon' 8,$$

where $0 \leqslant c \leqslant 2^{n-1} - 2$ (from the remarks), $\varepsilon = 0$ or 1, $\varepsilon' = 0$ or 1, $\varepsilon + \varepsilon' \leqslant 1$ and $0 \leqslant \alpha \leqslant (2^{n-2} - 4)/3$. Note that these are not the best bounds that could be used, but they are valid for n odd or even and are sufficient for what we wish to conclude. Indeed, again by a very rough calculus, we get cycle lengths down to at least $n2^n - (n-2)2^{n-1}$ which gives what we want since $n \geqslant 4$. □

5. Existence of small cycles in CCC_n

We first recall some well-known properties of CCC_n that we will use in some technical lemmata to construct small cycles by induction.

Fact 1. *CCC_{n+1} contains two disjoint induced subgraphs each one isomorphic to the graph obtained from CCC_n by deleting all edges between levels $n-1$ and 0.*

This results from the one-to-one mappings from $V(CCC_n)$ to $V(CCC_{n+1})$:

$$(\ell, x_0x_1\cdots x_{n-1}) \to (\ell, x_0x_1\cdots x_{n-1}0)$$

and

$$(\ell, x_0x_1\cdots x_{n-1}) \to (\ell, x_0x_1\cdots x_{n-1}1).$$

Fact 2. *For any edges $[(\ell,x),(\ell,x(\ell))]$ and $[(\ell,y),(\ell,y(\ell))]$, $0\leqslant\ell\leqslant n-1$, contained in level ℓ, there exists an automorphism of the cube-connected cycles CCC_n which sends vertices $(\ell,x),(\ell,x(\ell))$ on vertices $(\ell,y),(\ell,y(\ell))$, respectively, and do not change the level of any vertex.*

This property comes from the edge-transitivity property of $H(n)$. Thus, if CCC_n has a cycle of length l containing an edge in level ℓ, for some $\ell, 0\leqslant\ell\leqslant n-1$, then CCC_n contains a cycle of the same length through any edge contained in level ℓ.

Definition 5.1. We say that a cycle is of *type* 0 in CCC_n if it is a cycle of CCC_n which contains no C-edge between two levels $n-1$ and 0, but which contains at least two H-edges $[(n-1,x),(n-1,x(n-1))]$ and $[(n-1,y),(n-1,y(n-1))]$ in the level $n-1$. We denote by $\mathcal{T}_0(n)$ the set of the lengths of the cycles of type 0 in CCC_n.

Definition 5.2. Similarly, we say that a cycle is of *type* 1 in CCC_n if it is a cycle of CCC_n which contains exactly one C-edge between the levels $n-1$ and 0, but which contains at least two H-edges in the level $n-1$, these edges being not adjacent to the previous C-edge. We denote by $\mathcal{T}_1(n)$ the set of the lengths of the cycles of type 1 in CCC_n.

We first begin with the case $n=3$.

Lemma 5.3. *CCC_3 contains cycles of type 0 and lengths $8,12,14,16,20$ and cycles of type 1 and lengths $9,11,13,15,17,19,21$. It also contains cycles of lengths $3,10,18,22,23,24$ which are neither of type 0 nor of type 1.*

Proof. This result is not difficult to check. $\square$

Lemma 5.4. *If CCC_n contains a cycle of type 0 and length l, then CCC_{n+1} contains cycles of lengths $l, l+4, l+6$ and of type 0. If, furthermore, CCC_n contains a cycle of type 0 and length l', then CCC_{n+1} contains a cycle of length $l+l'+4$.*

Proof. Notice that, by Fact 1, a cycle of type 0 in CCC_n is also a cycle of type 0 in CCC_{n+1}. Consider a cycle $\mathscr{C}$ in CCC_n of length l and of type 0 as a subgraph of CCC_{n+1} using the one-to-one mapping

$$(\ell, x_0x_1\cdots x_{n-1}) \rightarrow (\ell, x_0x_1\cdots x_{n-1}0).$$

Let $[(n-1,x),(n-1,x(n-1))]$ be one H-edge of this cycle lying in the level $n-1$. We extend this cycle to a cycle of length $l+6$ in CCC_{n+1} by deleting the edge $[(n-1,x),(n-1,x(n-1))]$ and adding the path $[(n-1,x),(n,x),(n,x(n)),(n-1,x(n)),(n-1,x(n-1,n)),(n,x(n-1,n)),(n,x(n-1)),(n-1,x(n-1))]$. Notice that this new cycle has no C-edge between levels n and 0 and has two edges in level n.

In order to obtain a cycle of length $l+4$ from $\mathscr{C}$, assume, without loss of generality, that cycle $\mathscr{C}$ contains in level $n-1$ the two H-edges $[(n-1,x),(n-1,x(n-1))]$ and $[(n-1,y),(n-1,y(n-1))]$ and that vertices $(n-1,x),(n-1,y),(n-1,y(n-1)),(n-1,x(n-1)$ are placed in this order on $\mathscr{C}$. Delete from $\mathscr{C}$ the two H-edges $[(n-1,x),(n-1,x(n-1))]$ and $[(n-1,y),(n-1,y(n-1))]$. Doing so we obtain two paths: P_1 of extremities $(n-1,y),(n-1,x)$, P_2 of extremities $(n-1,y(n-1)),(n-1,x(n-1))$, the sum of the lengths of these paths being equal to $l-2$. By Fact 1, CCC_{n+1} contains a path P_2' isomorphic to P_2, disjoint from P_1, and of extremities $(n-1,x(n)),(n-1,y(n))$. A cycle of length $l+4$ is obtained by considering $P_1,(n,x),(n,x(n)),P_2',(n,y(n)),(n,y)$ (see an example in Fig. 3).

In order to obtain a cycle of length $l+l'+4$ in CCC_{n+1}, consider $\mathscr{C}$ and a cycle $\mathscr{C}'$ of length l' and type 0. The idea of the construction is similar to the previous one.

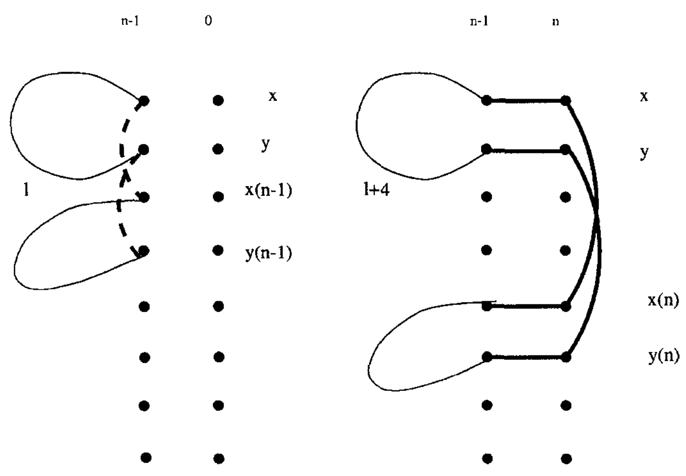

Fig. 3. Construction of a cycle of length $l+4$ in CCC_{n+1}.

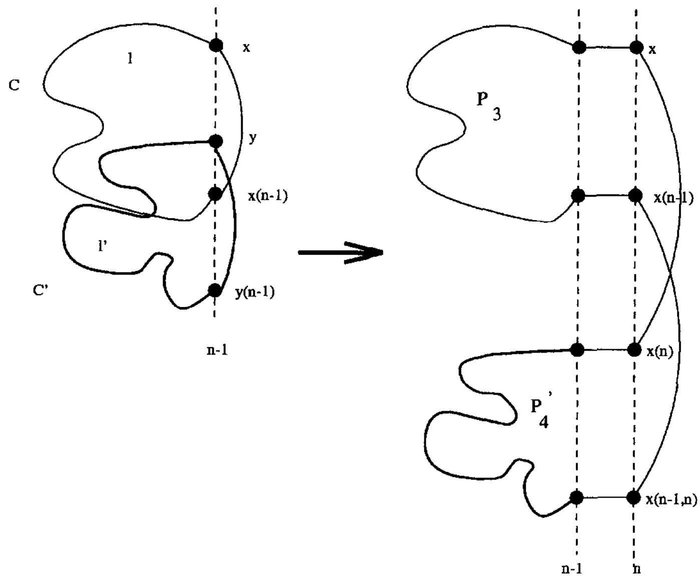

Fig. 4. Construction of a cycle of length $l + l' + 4$ in CCC_{n+1}.

Let $[(n-1,x),(n-1,x(n-1))]$ be an H-edge of $\mathscr{C}$ and $[(n-1,y),(n-1,y(n-1))]$ an H-edge of $\mathscr{C}'$. Denote by P_3, P_4 the paths obtained from $\mathscr{C}$, $\mathscr{C}'$, respectively, by deleting edges $[(n-1,x),(n-1,x(n-1))]$, $[(n-1,y),(n-1,y(n-1))]$, respectively. By Facts 1 and 2, CCC_{n+1} contains a path P_4' isomorphic to P_4, disjoint from P_3, and of extremities $(n-1,x(n)),(n-1,x(n-1,n))$. A cycle of length $l+l'+4$ in CCC_{n+1} is obtained by considering $P_3,(n,x),(n,x(n)),P_4',(n,x(n-1,n)),(n,x(n-1))$. $\square$

The proof is easier to understand from Fig. 4.

Proposition 5.5. *For any* $n \geqslant 3$, CCC_n *contains cycles of type* 0 *of all even lengths between 8 and* $3 \times 2^n - 4$, *except* 10 *and* $3 \times 2^n - 6$.

Proof. It follows directly from Lemma 5.4 by induction on n. By Lemma 5.3, $\mathscr{T}_0(3)$ contains $\{8,12,14,16,20\}$. Assume that, for a given $n \geqslant 3$, $\mathscr{T}_0(n)$ contains cycles of all even lengths between 8 and $3 \times 2^n - 4$ except 10 and $3 \times 2^n - 6$. Using Lemma 5.4, we first get that $\mathscr{T}_0(n+1)$ contains all cycle lengths $\mathscr{T}_0(n)$. Then because $\mathscr{T}_0(n)$ contains $3 \times 2^n - 4$ and all even values from 8 up to $3 \times 2^n - 8$ except 10, then $\mathscr{T}_0(n+1)$ contains cycle lengths $3 \times 2^n - 4 + 4 + l = 3 \times 2^n + l$ for $l = 0, 2, 8, 12, 14, \ldots s, 3 \times 2^n - 8$ and also $l = 3 \times 2^n - 4$. So it remains to show that $\mathscr{T}_0(n+1)$ contains cycles of lengths $3 \times 2^n - 6$, $3 \times 2^n - 2$, $3 \times 2^n + 4$, $3 \times 2^n + 6$ and $3 \times 2^n + 10$. But it is not difficult to get these values using Lemma 5.4 and starting with different values from $\mathscr{T}_0(n)$, since $3 \times 2^n - 6 = 3 \times 2^n - 12 + 6$, $3 \times 2^n - 2 = 3 \times 2^n - 8 + 6$, $3 \times 2^n + 4 = 3 \times 2^n - 8 + 8 + 4$, $3 \times 2^n + 6 = 3 \times 2^n - 10 + 8 + 4$ and $3 \times 2^n + 10 = 3 \times 2^n - 8 + 14 + 4$. $\square$

The following lemma is similar to Lemma 5.4, but for cycles of type 1.

Lemma 5.6. *If CCC_n contains a cycle of type* 1 *and length l', then CCC_{n+1} contains a cycle of type* 1 *for each length $l'+1, l'+7$. If, furthermore, CCC_n contains a cycle of type* 0 *and length l, then CCC_{n+1} contains a cycle of type* 1 *and length $l+l'+5$.*

Proof. Let $\mathscr{C}$ be a cycle of length l and of type 1 in CCC_n. Suppose that $\mathscr{C}$ contains the C-edge $[(n-1,x),(0,x)]$ between levels $n-1$ and 0 in CCC_n. It is sufficient to use Fact 1 and to replace this edge of CCC_n by the path in CCC_{n+1} of length 2 $[(n-1,x),(n,x),(0,x)]$ to obtain a path of length $l'+1$.

Any C-edge $[(n-1,x),(0,x)]$, $x \in Z_2^n$, used in a cycle of type 1 in CCC_n is extended to a path of length 2, $[(n-1,x),(n,x),(0,x)]$ in the corresponding cycle of CCC_{n+1}. Otherwise, in order to obtain cycle lengths $l'+7$ and $l+l'+5$, the constructions are the same as that of Lemma 5.4 which give a path of length $l+6$ and $l+l'+4$, respectively. □

Using Lemma 5.6 and Proposition 5.5, we get the following analogue of Proposition 5.5 concerning the existence of cycles of type 1.

Proposition 5.7. *For any $p \geqslant 2$, CCC_{2p} contains cycles of type* 1 *and all even lengths between $2p+6$ and $3 \times 2^{2p} + 2p - 6$.*

For any $p \geqslant 1$, CCC_{2p+1} contains cycles of type 1 *and of all odd lengths between $2p+7$ and $3 \times 2^{2p+1} + 2p - 5$.*

Proof. We proceed by induction on p. By Lemma 5.3, $\mathscr{T}_1(3)$ contains $\{9, 11, 13, 15, 17, 19, 21\}$. Let us assume that, for some $p \geqslant 2$, $\mathscr{T}_1(2p-1)$ contains all odd cycle lengths between $2p+5$ and $3 \times 2^{2p-1} + 2(p-1) - 5$.

Since, by Proposition 5.5, $\mathscr{T}_0(2p-1)$ contains all even cycle lengths between 8 and $3 \times 2^{2p-1} - 4$, except 10 and $3 \times 2^{2p} - 6$, then, using Lemma 5.6, $\mathscr{T}_1(2p)$ contains all even cycle lengths between $2p+6$ and $3 \times 2^{2p-1} - 4 + 3 \times 2^{2p-1} + 2p - 1 - 6 + 1 + 4$ which is equal to $3 \times 2^{2p} + 2p - 6$. Now, using Proposition 5.5 and Lemma 5.6, we obtain from the values in $\mathscr{T}_0(2p)$ and $\mathscr{T}_1(2p)$ that $\mathscr{T}_1(2p+1)$ contains all odd cycle lengths between $2p+7$ and $3 \times 2^{2p+1} + 2p + 1 - 6$. □

Before summarizing the results of this section, let us recall the following result proving that CCC_n does not contain some small cycles.

Proposition 5.8 (Heydemann et al. [5]). *Every non-fundamental cycle of CCC_n has length at least* 8. *Every non-fundamental cycle of CCC_5 has length 8 or at least* 11. *If $n \geqslant 6$, every non-fundamental cycle of CCC_n has length* 8 *or length at least* 12.

Proposition 5.8 shows, in particular, that, if $n \geqslant 5$, then $CCC(n)$ does not contain a cycle of length 10. Thus, the result of this section is the following proposition.

Proposition 5.9. *CCC_3 contains cycles of length 3 and all lengths between 8 and 24. For n even, $n \geqslant 4$, CCC_n contains cycles of all even lengths between 8 and $3 \times 2^n + n - 6$, except 10. For n odd, $n \geqslant 5$, CCC_n contains cycles of all odd lengths between $n+6$ and $3 \times 2^n + n - 6$ and cycles of all even lengths between 8 and $3 \times 2^n - 4$ except 10 and $3 \times 2^n - 6$.*

6. Conclusions

The proof of Theorem 1.1 is now evident. First CCC_3 contains cycles of all possible lengths except 4, 5, 6, 7. Let us recall that fundamental cycles are of length n. Then for any n, $n \geqslant 4$, other cycle lengths are obtained as follows. Even length values l are covered by

- Proposition 5.9, for $8 \leqslant l \leqslant 3 \times 2^n - 8$ except $l = 10$,
- Proposition 3.6, for $3 \times 2^n - 12 \leqslant l \leqslant (n-1)2^n + 12$,
- Proposition 4.14, for $(n-1)2^n \leqslant l \leqslant n2^n$ except possibly $l = n2^n - 2$.

For $n = 4$, these results give cycles of all even lengths between 8 and $64 = 4 \times 2^4$ excepted 10 (in this case $n2^n - 2 = n2^n - (n-2) = 62$ is obtained). But it is easy to find in CCC_4 a cycle of length 10:

$$[(0,0^4),(3,0^4),(2,0^4),(1,0^4),(1,0100),(0,0100),$$
$$(0,1100),(1,1100),(1,1000),(0,1000)].$$

Thus, CCC_4 contains cycles of all even lengths excepted 6.

For n odd, $n \geqslant 5$, odd length values l are covered by

- Proposition 5.9, for $n + 6 \leqslant l \leqslant 3 \times 2^n - 1$,
- Proposition 3.6, for $3 \times 2^n - 19 \leqslant l \leqslant (n-1)2^n + 19$,
- Proposition 4.14, for $l = n2^n - n + 2$ and $(n-1)2^n + 1 \leqslant l \leqslant n2^n - n - 2$.

Thus, in this article we have proved that the cube-connected cycles graph CCC_n is almost pancyclic. Let us now consider the missing values of cycle length. By Proposition 5.8 and Theorem 1.1, for CCC_5 it remains to study the existence of cycles of lengths $5 \times 2^5 - 5 = 155, 5 \times 2^5 - 2 = 158, 5 \times 2^5 - 1 = 159$.

The possible values of length cycle l which are not covered by our constructions for $n \geqslant 6$ are $l = n2^n - 2$ and, in case n odd, odd lengths l, $l \geqslant 13$, for

- $l \leqslant n - 2$, $l = n + 2$, $l = n + 4$,
- $l = n2^n - n$,
- $n2^n - n + 4 \leqslant l \leqslant n2^n - 1$.

Notice that, since CCC_n is a subgraph of the butterfly graph [4], our results apply also to this graph.

Bondy "meta-conjectured" in [3] that almost any non-trivial condition on a graph which implies that the graph is hamiltonian also implies that the graph is pancyclic. One can ask whether like CCC_n every Cayley graph on a wreath product is almost pancyclic or even pancyclic.

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